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ON THE LOWER PORTION OF THE SPECTRUM OF NATURAL AXISYMMETRIC VIBRATIONS OF A THIN ELASTIC SHELL OF REVOLUTION

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N. V. KHAR'KOVA
(Moscow)
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Conditions are presented for which the lower part of the spectrum of the membrane problem consists of an infinite series of eigenvalues converging to the lower bound of the continuous spectrum. It is shown that boundary layer theory [1] is applicable to this portion of the spectrum and the first approximation is obtained for the eigenvalues.

The equations of natural axisymmetric vibrations of a thin elastic shell of revolution are [2, 3]:

$$-\frac{d}{ds}\left(\frac{1}{B(s)}\frac{dBu}{ds}\right)-\left(\frac{1-\sigma}{R_1(s)R_2(s)}\right)u+\left(\frac{1}{R_1}+\frac{\sigma}{R_2}\right)\frac{dw}{ds}+\frac{d}{ds}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)w=\lambda u \qquad (0.1)$$

$$-\left(\frac{1}{R_1}+\frac{\sigma}{R_2}\right)\frac{du}{ds}-\left(\frac{\sigma}{R_1}+\frac{1}{R_2}\right)\frac{B'}{B}u+\left(\frac{1}{R_1^2}+\frac{2\sigma}{R_1R_2}+\frac{1}{R_2^2}\right)w+\frac{h^2}{12}\frac{1}{B}\frac{d}{ds}\left(B\frac{d}{ds}\left(B\frac{dw}{ds}\right)\right)=\lambda w$$

Here the parameter s is the arclength of a meridian of the middle surface measured from some fixed point, $B(s)$ is the distance between a variable point on the meridian and the axis of revolution. The projections of displacement of the middle surface point in the directions of the meridian and of the normal to the surface and denoted by $u(s)$ and $w(s)$. For the principal radii of curvature we have

$$R_1^{-1}=-B''(1-(B')^2)^{-1/2}, \qquad R_2^{-1}=(1-(B')^2)^{1/2}B^{-1}$$

The spectral parameter λ is proportional to the square of the vibrations frequency, the small parameter h is the relative shell thickness, and σ is Poisson's ratio. The coefficients of (0.1) are assumed sufficiently smooth.

Let us bound the shell by two parallels $s=s_1$ and $s=s_2$, and let us take the following boundary conditions

$$u(s_1)=u(s_2)=w(s_1)=w(s_2)=w'(s_1)=w'(s_2)=0 \qquad (0.2)$$

Besides the system (0.1), the membrane system of equations ($h = 0$)

$$-\frac{d}{ds}\left(\frac{1}{B}\frac{dBu}{ds}\right)-\left(\frac{1-\sigma}{R_1R_2}\right)u+\left(\frac{1}{R_1}+\frac{\sigma}{R_2}\right)\frac{dw}{ds}+\frac{d}{ds}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)w=\lambda u \tag{0.3}$$

$$-\left(\frac{1}{R_1}+\frac{\sigma}{R_2}\right)\frac{du}{ds}-\left(\frac{\sigma}{R_1}+\frac{1}{R_2}\right)\frac{B'}{B}u+\left(\frac{1}{R_1^2}+\frac{2\sigma}{R_1R_2}+\frac{1}{R_2^2}\right)w=\lambda w$$

with the boundary conditions

$$u(s_1)=u(s_2)=0 \tag{0.4}$$

will be considered.

After the researches contained in [4] and [3] it became clear that the determination of the natural shell vibrations frequencies belonging to a range of variation of the function

$$\varphi_1(s)=\frac{1-\sigma^2}{R_2^2(s)} \qquad (s_1\leqslant s\leqslant s_2) \tag{0.5}$$

by asymptotic methods causes difficulties because of the presence of a turning point in the system (0.3). Let $[\alpha, \beta]$ denote the range of variation of the function (0.5). As has been shown by Lidskii and the author in [5], this range belongs to the continuous spectrum of the problem (0.3), (0.4).

Conditions are mentioned in Sect. 1 for which there exists an infinite series of eigenvalues of the boundary value problem (0.3), (0.4) which converges from below to the point α, and also given is the condition for which there exists an infinite series converging to the point β from above (*). A continuous spectrum, as well as an infinite series of eigenvalues, converging to a finite point cannot exist in bending theory ($h \neq 0$), but any finite number of eigenvalues similar to the membrane values, can be obtained on the portion $\lambda_n < \alpha$ for sufficiently small h. It is shown in Sect. 2 that the eigenvalues satisfying the inequality $\lambda_n < \alpha$ asymptotic expansions in powers of $h^{1/4}$ are valid (the conditions of degeneration regularity in the sense stated by the authors of [1] are satisfied).

1. Membrane case. Let us replace the system (0.3) by one equation. To do this, let us find w from the second equation and let us substitute it into the first equation. If λ does not belong to the domain of values of the function

$$\varphi_2(s)=\frac{1}{R_1^2(s)}+\frac{2\sigma}{R_1(s)R_2(s)}+\frac{1}{R_2^2(s)} \tag{1.1}$$

then this can always be done. We obtain

$$(\lambda-\varphi_1(s))\frac{d^2u}{ds^2}+b(s,\lambda)\frac{du}{ds}+c(s,\lambda)u=0 \tag{1.2}$$

$$b(s,\lambda)=b_1(s)\lambda+b_2(s)+\frac{b_3(s)}{\lambda-\varphi_2(s)} \tag{1.3}$$

$$c(s,\lambda)=\lambda^2+c_1(s)\lambda+c_2(s)+\frac{c_3(s)}{\lambda-\varphi_2(s)} \tag{1.4}$$

Here $b_i(s)$ and $c_i(s)$ ($i = 1, 2, 3$) are some smooth functions independent of λ. Let us recall that

$$\alpha=\inf\varphi_1(s) \qquad (s\in[s_1, s_2]) \tag{1.5}$$

If $\lambda = \alpha$, then (1.2) becomes singular. For simplicity, let us assume that $\inf \varphi_1(s)$

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*) Such a series was noted earlier in the case of a cylinder and a sphere [6, 7].

is achieved in (1.5) at one point $s = s_0$. For definiteness, let $s_0 \neq s_1$. Let us form the Cauchy problem (1.2) at the left endpoint

$$u(s_1, \lambda) = 0, \qquad u'(s_1, \lambda) = 1 \quad (\lambda < \alpha) \tag{1.6}$$

Let us first establish the following auxiliary proposition.

Lemma 1.1. For $\lambda = \alpha > 0$ let the solution $u(s, \lambda)$ of the problem (1.6) for (1.2) have an infinite number of zeros in the open interval $[s_1, s_0)$. Then an infinity of eigenvalues of the problem (0.3), (0.4) is located below α.

Proof. Let N be an arbitrary natural number. Let us select δ so small that the function $u(s, \alpha)$ would have $N + 1$ zeros in the segment $[s_1, s_0 - \delta]$. Afterwards, let us select $\lambda < \alpha$ so slightly different from α that $u(s, \lambda)$ would also vanish $N + 1$ times on the segment $[s_1, s_0 - \delta)$. This can be done because of the theorem on the continuous dependence of the solution of (1.6) on the parameter λ.

Let us diminish λ. We note that as $\lambda \to -\infty$ the solution $u(s, \lambda)$ has no zeros at all within the segment $[s_1, s_2]$, as follows from (1.2)–(1.4) taking account of the Sturm theorem (*). Since the zeros of the solution of (1.6) are not multiple for $\lambda < \alpha$ (by virtue of the uniqueness theorem) and are continuous functions of λ, then as λ diminishes they pass through the point s_2 (they cannot reach s_1 since $u(s,\lambda) \not\equiv 0$). Therefore, no less than N eigenvalues of the problem (0.3), (0.4) exist. Since N is arbitrary, Lemma 1.1 is proved.

Note 1.1. From the proof of the Lemma it follows that if there are N zeros in the interval (s_1, s_0) for the Cauchy solution at $\lambda = \alpha$ then below $\lambda = \alpha$ at least N eigenvalues are located.

Note 1.2. An analogous lemma can be proved even when the set of points s_0 is arbitrary. The Cauchy problem must then be posed at the point s', where $\varphi_1(s') > \alpha$.

It is assumed below that $s_0 = 0$.

Lemma 1.2. In the neighborhood of the point $s_0 = 0$ let (1.2) be

$$L(s^2 + O(s^3))u'' + 2L(s + O(s^2))u' + (M + O(s))u = 0 \tag{1.7}$$

where the terms $O(s^i)$ admit of formal differentiation. Then in order for the nontrivial solution to have an infinity of zeros, it is necessary and sufficient that

$$L^2 - 4LM < 0$$

Proof. The proof is analogous to that in [8].

Now, in the neighborhood of the point $s = 0$ let the function $B(s)$ have the Taylor expansion

$$B(s) = B_0 + B_1 s + {}^1/_2 B_2 s^2 + {}^1/_6 B_3 s^3 + \ldots \tag{1.8}$$

Here $B_0 > 0$, $|B_1| \leqslant 1$ and it can always be considered that $B_1 \geqslant 0$. It is easy to verify that the function $\varphi_1(s)$ can have an extremum at the point $s = 0$ either for $B_1 = 0$ or

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*) Indeed the substitution

$$u = v \exp\left(-\frac{1}{2}\int \frac{b(s, \lambda)}{\lambda - \varphi_1(s)}\, ds\right)$$

results in the binomial equation

$$v'' + p(s, \lambda)\, v = 0, \qquad p(s, \lambda) = \lambda + O(1) < 0$$

for $B_2 = (B_1^2 - 1) B_0^{-1}$. The first condition is evidently satisfied if $s = 0$ is a stationary point of the meridian, and for $B_1 \neq 1$ the second condition agrees with the condition $R_1 = R_2$, i.e. the corresponding point is umbilical. Having investigated these cases we obtain two theorems on the infiniteness of the lower series of eigenvalues.

Theorem 1.1. In the expansion (1.8) let

$$B_1 = 0, \qquad -1 \leqslant B_0 B_2 \leqslant 0 \tag{1.9}$$

$$4\sigma^2 - (12\sigma - 1)B_0B_2 + 9\,(B_0B_2)^2 > 0$$

and let the function $\varphi_1(s)$ have a minimum at $s = 0$ which is absolute: $\varphi_1(0) = \alpha > 0$, then there exists an infinite series of eigenvalues of the problem (0.3), (0.4) which converges to the point α from below.

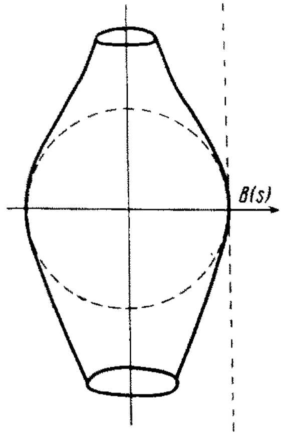

Fig. 1

Note 1.3. The second condition in (1.9) means geometrically that the function $B(s)$ has a maximum at the point $s = 0$ and the radius of the circle adjoining the meridian at this point is not less than the distance to the axis of revolution (Fig. 1). A sphere and cylinder are two extreme cases of such a shell for which B_0B_2 equal -1 and 0, respectively.

Note 1.4. For $\sigma > {}^1/_{24}$ the third condition of (1.9) is satisfied as long as the first two are satisfied.

Proof of Theorem 1.1. By direct computation it can be verified that if $B_0B_2 \neq 0, -1$, then in the neighborhood of the point $s = 0$ Eq. (1.2) has the form (1.7) for $\lambda = \alpha$ and

$$L = -[(B_0B_2)^2 + B_0B_2], \qquad M = [2\,(B_0B_2)^2 - 3\sigma B_0B_2 + \sigma^2]$$

There remains just to apply Lemmas 1.2 and 1.1. If B_0B_2 equals 0 or -1, then $L = 0$ and the proof becomes more complex. The appropriate investigation is omitted here.

Theorem 1.2. Let $0 < B_1 < 1$ in the expansion (1.8) and

$$B_2 = (B_1^2 - 1)B_0^{-1}, \qquad B_3 \leqslant B_1\,(B_1^2 - 1)B_0^{-2}$$

For $s = 0$ let the function $\varphi_1(s)$ have a minimum which is absolute, $\varphi_1(0) = \alpha > 0$. Then there exists an infinite series of eigenvalues converging to the point α from below.

The proof is analogous to the proof of Theorem 1.1. For $\lambda = \alpha$ Eq. (1.2) has the form (1.7) in the neoghborhood of the point $s = 0$ with

$$\begin{aligned} L &= -B_1B_3B_0^{-2} - B_1^2\,(1 - B_1^2)\,B_0^{-4} \\ M &= L + (2 + 3\sigma + \sigma^2)\,(1 - B_1^2)^2 B_0^{-4} \end{aligned} \tag{1.10}$$

If $L \neq 0$, then by applying Lemmas 1.2 and 1.1 we obtain the assertion of the Theorem. When $L = 0$ an additional investigation, omitted here, is made.

Note 1.5. If the function $\varphi_1(s)$ reaches a minimum on the boundary with a nonzero derivative, then by analogous methods it can be shown that there cannot be an infinity of zeros in the solution of the corresponding singular equation. It follows from Theorem 1.2 (see [5]) that there can be only a finite number of eigenvalues lying below α.

Note 1.6. An analysis of the equations obtained for $\lambda = \beta = \sup\varphi_1(s)$ shows that an infinity of zeros in the nontrivial solution exists only when the point s_0 with $\varphi_1(s_0) = \beta$ is an umbilical, and $-3L-4M < 0$ (L and M are the same as in (1.10)). If $\inf \varphi_2(s) > \sup \varphi_1(s)$, then this condition is sufficient for the existence of an infinite series of eigenvalues of the problem (0.3), (0.4) which converges to the point β from above.

2. Approximation of eigenvalues and eigenfunctions of the bending problem. To seek the eigenvalues of the bending problem $\lambda_k < \alpha$ the asymptotic method stated in [1] can be applied, where Theorem 13 is essentially utilized in the proof. Let us just note the changes which should be inserted in the proof in connection with the fact that a system is considered rather than one equation. In order not to complicate the exposition, we shall limit ourselves just to the first approximation for the eigennumbers and eigenvectors.

Let us rewrite the system (0.1) in the form

$$\begin{gathered}-u'' + a_1u' + a_0u + b_1w' + b_0w = \lambda u \qquad (2.1)\\ c_1u' + c_0u + d_0w + \varepsilon^4(w^{IV} + d_3w''' + d_2w'' + d_1w') = \lambda w, \qquad \varepsilon^4 = {}^1/_{12}\, h^2\end{gathered}$$

The left sides of (0.3) and (2.1) give the linear operators L_0 and $L_\varepsilon = L_0 + \varepsilon^4 L_1$, which it is essential to examine in the space of function pairs $f = (u, w)$ where the scalar product is introduced by means of the formula

$$(f_1, f_2) = \int_{s_1}^{s_2} (u_1\bar{u}_2 + w_1\bar{w}_2)\, B\, ds$$

It is easy to verify that the operator L_0 is symmetric and positive definite under the boundary conditions (0.4), and the operator L_ε is symmetric under the boundary conditions (0.2). The same letters will denote their closures.

Let $s_1 = 0$ and in the neighborhood of the point 0 let all the coefficients of (2.1) be expanded in Taylor series as for example:

$$a_1(s) = a_1{}^\circ + a_1{}^1 s + a_1{}^2 s^2 + \ldots$$

Let us set

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \ldots$$

Let us transfer all the terms in (2.1) to the left, and make the change of variable $s = \varepsilon t$ We note that it is more convenient to consider the unknown function to be $\varepsilon^{-1}u$ rather than u . Ordering terms in powers of ε we obtain

$$\varepsilon^{-1}\sum_{i=0}^{2} \varepsilon^i M_i{}^1(\varepsilon^{-1}u, w) + \varepsilon^2 M_3{}^1(\varepsilon^{-1}u, w) = 0$$

$$\sum_{i=0}^{2} \varepsilon^i M_i{}^2(\varepsilon^{-1}u, w) + \varepsilon^3 M_3{}^2(\varepsilon^{-1}u, w) = 0$$

The left side of these formulas yields the partition of the operator $L_\varepsilon - \lambda_\varepsilon I$ in the neighborhood of the point $s = 0$ which plays the same part as (2.9) from [1]. Here the operators $M_0{}^1$ and $M_0{}^2$ with constant coefficients are

$$M_0{}^1(\varepsilon^{-1}u, w) = -(\varepsilon^{-1}u)_t'' + b_1{}^\circ w_t'$$

$$M_0{}^2(\varepsilon^{-1}u, w) = -b_1{}^\circ(\varepsilon^{-1}u)_t' + d_0{}^\circ w + w_t{}^{IV} - \lambda_0 w$$

The coefficients of the operators M_i^1 and M_i^2 $(i = 1, 2)$ are polynomials of degree no greater than i, and the coefficients of the operators M_3^1 and M_3^2 are third degree polynomials multiplied by bounded functions. The system

$$M_0^1(\varepsilon^{-1}u, w) = 0, \qquad M_0^2(\varepsilon^{-1}u, w) = 0$$

has the following solutions

$$z_i^\circ = (u_i^\circ, w_i^\circ), \qquad \varepsilon^{-1}u_i^\circ = b_1^\circ\rho_i^{-1}e^{\rho it}, \qquad w_i^\circ = e^{\rho it}, \qquad t = s/\varepsilon \tag{2.2}$$

for $\lambda < (1 - \sigma^2)R_2^{-2}(0)$ where the ρ_i $(i = 1, 2, 3, 4)$ are fourth power roots of

$$(\lambda_0 - (1 - \sigma^2)R_2^{-2}(0))$$

Two other solutions of this system are obtained for $\rho_5 = \rho_6 = 0$. Two roots ρ_1 and ρ_2 lie in the left half-plane, hence the degeneration is regular in the sense of [1]. An analogous partition of the operator can be constructed at the right endpoint and we see that the right endpoint is also regular.

The first approximation for the eigenfunctions is sought in the form

$$f_\varepsilon^1 = f_0 + \varepsilon f_{10} + z_1 + \varepsilon z_2 + \varepsilon^2 z_3 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3$$

Here the vectors f_0 and f_{10} are solutions of the degenerate system, $z_i = (u_i, w_i)$ are solutions of boundary layer type and α_i are corrections. These vectors are constructed as follows. In the expression

$$L_\varepsilon f_\varepsilon^1 - (\lambda_0 + \lambda_1\varepsilon) f_\varepsilon^1 \tag{2.3}$$

we equate terms in the smallest powers of ε to zero. We obtain five systems of equations

$$L_0 f_0 - \lambda_0 f_0 = 0 \tag{2.4}$$

$$M_0^1(\varepsilon^{-1}u_1, w_1) = 0, \qquad M_0^2(\varepsilon^{-1}u_1, w_1) = 0 \tag{2.5}$$

$$(L_0 - \lambda_0) f_{10} - \lambda_1 f_0 = \lambda_0\alpha_1 - L_0\alpha_1 \tag{2.6}$$

$$M_0^1(\varepsilon^{-1}u_2, w_2) = -M_1^1(\varepsilon^{-1}u_1, w_1), \qquad M_0^2(\varepsilon^{-1}u_2, w_2) = -M_1^2(\varepsilon^{-1}u_1, w_1) \tag{2.7}$$

$$M_0^1(\varepsilon^{-1}u_3, w_3) = -M_1^1(\varepsilon^{-1}u_2, w_2) - M_2^1(\varepsilon^{-1}u_1, w_1) \tag{2.8}$$

$$M_0^2(\varepsilon^{-1}u_3, w_3) = -M_1^2(\varepsilon^{-1}u_2, w_2) - M_2^2(\varepsilon^{-1}u_1, w_1)$$

The vector f_0 is the solution of the degenerate system (2.4) under the boundary conditions (0.4). The vector $z_1 = (u_1, w_1)$ is sought in the neighborhood of the point 0 in the form of a linear combination of the solutions z_1° and z_2° from (2.2) so that the sum $f_0 + z_1$ would satisfy the boundary conditions on the component w from (0.2) at the left endpoint. We find the vector z_1 in the neighborhood of the right endpoint analogously, and we match them exactly as in [1] by using infinitely differentiable cutoff functions. We find the correction α_1 in the form $\alpha_1 = (\alpha_{1u}, 0)$, where α_{1u} is a zero degree polynomial such that the sum $f_0 + z_1 + \varepsilon\alpha_1$ satisfies all the boundary conditions (0.2) at the left endpoint. We then construct the correction at the right endpoint and match them. Afterwards we find the solution f_{10} of the system (2.6) which satisfies the boundary conditions (0.4). This solution exists ([1], p. 110, Note c) if

$$\lambda_1 = -(\lambda_0\alpha_1 - L_0\alpha_1, f_0) \tag{2.9}$$

A solution z_2 of the system (2.7) can be found such that its components are the product of first degree polynomials in t by the exponential of the boundary layer (2.2), and the vector z_2 is such that $f_{10} + z_2$ satisfies conditions on w from (0.2) at the left endpoint. The vector z_2, exactly as z_1, is matched at the two ends and the correction α_2 is found.

The vectors z_3 and α_3 are sought analogously to z_2 and α_2 with the sole difference that z_3 must satisfy the boundary conditions on w from (0.2).

Now, if the found vectors are substituted into (2.3), we obtain

$$L_\varepsilon f_\varepsilon^{\,1} - (\lambda_0 + \lambda_1\varepsilon) f_\varepsilon^{\,1} = \varepsilon^2 (-\lambda_1 f_{10} + (L_0 - \lambda_0 I)\,\alpha_2 - \lambda_1\alpha_1 + Z) + \varepsilon^3 Y \qquad (2.10)$$

In the neighborhood of the left endpoint the vector Z has the components

$$Z_u = M_1^{\,1}(z_3) + M_2^{\,1}(z_2) + M_3^{\,1}(z_1), \qquad Z_w = 0$$

The norm of the vector Y is bounded: $\|Y\| \leqslant M$. This can be seen by writing all the terms in Y. Let us use the notation

$$\|-\lambda_1 f_{10} + (L_0 - \lambda_0 I)\,\alpha_2 - \lambda_1\alpha_1 + Z\| = N$$

Then for small ε it follows from (2.10) that

$$\|L_\varepsilon f_\varepsilon^{\,1} - (\lambda_0 + \lambda_1\varepsilon) f_\varepsilon^{\,1}\| \leqslant \varepsilon^2 2N \qquad (2.11)$$

Reasoning in the same manner as in the proof of Theorem 13 from [1], we obtain the following theorem.

Theorem 2.1. In the space of the function pairs (u, w) let be defined the operators L_0, by using (0.3) and (0.4), and $L_\varepsilon = L_0 + \varepsilon^4 L_1$, by using (2.1) and (0.2), and let λ_0 be the eigenvalue of the operator L_0 such that

$$\lambda_0 < \alpha = \inf\,(1 - \sigma^2) R_2^{-2}(s)$$

Then there exists an eigenvalue λ_ε of the operator L_ε with the asymptotic representation

$$\lambda_\varepsilon = \lambda_0 + \lambda_1\varepsilon + O(\varepsilon^2)$$

and the corresponding eigenfunction of the form

$$f_\varepsilon = f_0 + \varepsilon f_{10} + z_1 + \varepsilon z_2 + \varepsilon^2 z_3 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3 + \varepsilon^2 y_2$$

The terms f_0, f_{10}, z_i and α_i are here constructed as has been described. The vector y_2 has a bounded norm: $\|y_2\| = O(1)$. For λ_1 the following formula is valid: (*)

$$\lambda_1 = \sqrt{2}\, B(s_1)\, w_0^{\,2}(s_1) \left(\frac{1-\sigma^2}{R_2^{\,2}(s_1)} - \lambda_0\right)^{3/4} + \\ + \sqrt{2}\, B(s_2)\, w_0^{\,2}(s_2) \left(\frac{1-\sigma^2}{R_2^{\,2}(s_2)} - \lambda_0\right)^{3/4}$$

where $w_0(s)$ is a component of the vector f_0.

For small ε the difference $\lambda_\varepsilon - (\lambda_0 + \lambda_1\varepsilon) = O(\varepsilon^2)$ admits the estimate

$$|\lambda_\varepsilon - (\lambda_0 + \lambda_1\varepsilon)| < \varepsilon^2 3N$$

The formula for λ_1 is obtained directly from (2.9). The last estimate follows from (2.11) and the following estimate ([1], p. 114):

$$|\lambda_\varepsilon - (\lambda_0 + \lambda_1\varepsilon)| < \varepsilon^2 \frac{\|g\|}{\|f_\varepsilon^{\,1}\|}, \qquad g = \varepsilon^{-2}(L_\varepsilon f_\varepsilon^{\,1} - (\lambda_0 + \lambda_1\varepsilon) f_\varepsilon^{\,1})$$

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*) After the paper had been readied for the printer, the author learned that Tovstik had independently derived an analogous formula.

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ELECTROMECHANICAL VIBRATIONS OF CENTROSYMMETRIC CUBIC CRYSTAL PLATES

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R. D. MINDLIN
(Columbia University, New York, U. S. A.)

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Introduction. According to the classical theory of piezoelectricity, there can be no piezoelectric effect in centrosymmetric crystals. Consequently the theory has it that vibrations of a centrosymmetric crystal plate cannot be excited, for example, by applying an alternating voltage drop between electrodes on the opposing faces of the plate. This conclusion is a direct result of the assumption, in the theory, that the stored energy of deformation and polarization is a function of the strain and polarization only [1]. Hence the only possible electromechanical interaction energy is the product of a second rank tensor (strain) and a first rank tensor (polarization) – with a third rank material coefficient (a piezoelectric constant). Since there are no third rank centrosymmetric tensors, there is no piezoelectric effect in centrosymmetric materials.

There is reason to believe, however, that the stored energy of deformation and